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Functional Equations in the Theory of Dynamic Programming. XIV: Upper and Lower Bounds for Solutions of Nonlinear Partial Differential Equations*

RICHARD BELLMAN

Departments of Mathematics, Electrical Engineering, and Medicine, University of Southern California

1. INTRODUCTION

The theory of dynamic programming associates a nonlinear functional equation with a multistage decision process, or a process which can be interpreted in these terms [1]. The optimal policy is determined by the solution of the functional equation and, in turn, determines the solution. This equivalence between polices and functions is important in many ways. In particular, the method of "approximation in policy space" provides a straightforward way of obtaining bounds on the solution of the functional equation, upper bounds if a minimization process, lower bounds if a maximization process. This is one of the basic properties of quasilinearization [2], [3].

More difficult, and of correspondingly more significance, is the task of obtaining both upper and lower bounds. In this paper, we will indicate how to accomplish this for certain important classes of partial differential equations associated with variational problems which may be considered to arise from the theory of derterministic control processes.

To simplify the presentation, we shall restrict ourselves to the nonlinear partial differential equation

$$f_T = 2g(c) - \frac{f_c^2}{4}, \qquad f(c, 0) = h(c), \tag{1}$$

connected with the functional

$$J(u) = \int_0^T (u'^2 + 2g(u))\, dt + h(u(T)). \tag{2}$$

Analogous results can be obtained for more general scalar functionals and for the multidimensional case without the introduction of any new ideas.

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409/27/3-1

2. Multistage Decision Process of Continuous Deterministic Type

If we regard the minimization of $J(u)$ as a multistage decision process of continuous deterministic type, we readily obtain a partial differential equation for the function

$$f(c, T) = \min_u J(u), \tag{1}$$

where u is subject to the initial condition $u(0) = c$. This partial differential equation, which is that given in (1.1), can be obtained via the calculus of variations by an appeal to Hamilton–Jacobi theory. What is significant about the dynamic programming derivation, apart from its immediacy and simplicity [4], [5], is that the derivation of the equation yields the preliminary form

$$f_T = \min_v [v^2 + 2g(c) + vf_c], \qquad f(c, 0) = h(c). \tag{2}$$

Here $v = v(c, T)$ is the missing initial condition $u'(0)$ in the Euler equation, and also represents the optimal policy in the multistage decision process.

The question of the domain of validity of (2) is a difficult one under general assumptions concerning g and h. We shall assume that g and h are strictly convex in c which means that the minimum of $J(u)$ exists for all c and $T > 0$ where u is constrained by the conditions

$$u(0) = c, \qquad u' \in L^2(0, T). \tag{3}$$

The minimizing function is uniquely specified by the Euler equation [5]. This means that f satisfies (2) for all c and $T > 0$.

3. Upper Bounds

The identification of the solution of (1.1) with the function defined by (2.1) yields the first method of obtaining an upper bound. Let u_1 be an admissible function, satisfying (2.3). Then

$$f(c, T) \leqslant \int_0^T (u_1'^2 + 2g(u_1))\, dt_1 + h(u_1(T)). \tag{1}$$

For example, $u_1 = c$ is a candidate, yielding

$$f(c, T) \leqslant 2Tg(c) + h(c). \tag{2}$$

This corresponds to an approximation in function space. Alternatively, we can approximate in policy space, using (2.2). Let $v_1(c, T)$ be any function of c and T and let $f_1(c, T)$ be determined as the solution of the *linear* equation

$$(f_1)_T = v_1^2 + 2g(c) + v_1(f_1)_c, \qquad f_1(c, 0) = h(c). \tag{3}$$

Then

$$f(c, T) \leqslant f_1(c, T), \tag{4}$$

for $T \geqslant 0$.

This is a consequence of the monotonicity in the Collatz sense of the linear partial differential equation

$$u_T = a(c, T) + b(c, T)\, u_c, \qquad u(c, 0) = h(c). \tag{5}$$

Furthermore, if we write $f_1 = f_1(v_1, c, T)$, we have the representation

$$f = \min_{v_1} f_1(v_1, c, T). \tag{6}$$

See [2, 3, 6] for further results and references.

A representation in the form of (2.2) can be applied profitably to the equation

$$f_T = \varphi(f_c), \tag{7}$$

which has been extensively investigated in recent years. For a representation of the solution in the form shown in (6), due to Bellman–Lax, see [7]. For a detailed investigation of equations such as (1.1) using the associated variational problem, see [8], [9].

In what follows we wish to present some methods for obtaining lower bounds. For the scalar and matrix Riccati differential equations, ordinary differential equations, direct methods are available; [10], [11].

4. Nonlinear Monotone Equations

In passing, let us note that the representation in (2.1), namely,

$$f(c, T) = \min_u \left[\int_0^T (u'^2 + 2g(u))\, dt + h(u(T))\right], \tag{1}$$

permits us to deduce monotonicity properties for f as operations on g and h. If we write

$$f(c, T) = f(c, T; g, h), \tag{2}$$

then it is clear that

$$f(c, T; g_1, h_1) \geqslant f(c, T; g_2, h_2), \tag{3}$$

if $g_1 \geqslant g_2$, $h_1 \geqslant h_2$.

5. Duality

A systematic approach for obtaining lower bounds consists of transforming the original variational problem requiring a minimum into a new variational problem involving a maximum. Thus, we want to construct a new functional $K(v)$ with the property that

$$\min_u J(u) = \max_v K(v). \tag{1}$$

One way of doing this is to use a Legendre transformation, as was first indicated by Friedrichs; see [12]. We shall pursue a route which permits us more flexibility. It will yield the Friedrichs transformation, and many additional results as well.

6. The Case $h(c) = 0$

Let us begin with the case where $h(c) = 0$. Our first procedure hinges on the simple result

$$u'^2 = \max_v (2u'v - v^2). \tag{1}$$

Then

$$\int_0^T (u'^2 + 2g(u))\, dt = \max_v \left[\int_0^T (2u'v - v^2 + 2g(u))\, dt\right]. \tag{2}$$

Hence, for any function $v \in L^2(0, T)$, we have

$$J(u) = \int_0^T (u'^2 + 2g(u))\, dt \geqslant \left[\int_0^T (2u'v - v^2 + 2g(u))\, dt\right], \tag{3}$$

whence

$$\min_u J(u) \geqslant \min_u \left[\int_0^T (2u'v - v^2 + 2g(u))\, dt\right]. \tag{4}$$

To carry through the minimization with respect to u, we integrate by parts,

$$\int_0^T 2u'v\, dt = 2uv\Big]_0^T - \int_0^T 2uv'\, dt. \tag{5}$$

Let us restrict v by the condition $v(T) = 0$, since the maximizing v in (1) is equal to u' and $u'(T) = 0$ for the minimizing function. The right-hand side of (4) then becomes

$$\min_u \left[-2cv(0) + \int_0^T (2g(u) - 2uv')\, dt - \int_0^T v^2\, dt\right]. \tag{6}$$

If we now introduce the Fenchel transform [13],

$$h(w) = \min_z [g(z) - zw], \tag{7}$$

and perform the u-minimization in (6), we obtain the following result:

$$\min_u J(u) \geqslant \left[-2cv(0) + \int_0^T [h(v') - v^2]\, dt\right] = K(v). \tag{8}$$

Hence,

$$\min_u J(u) \geqslant \max_v K(v). \tag{9}$$

The constraints on v are $v(T) = 0$, $h(v') \in L(0, T)$. It can be shown by direct calculation that equality is actually attained in (9). A key observation is that h is concave.

Rayleigh–Ritz techniques can now be employed to provide accurate upper and lower bounds.

7. Use of the Convexity of g

An alternate procedure employs the convexity of g. We begin with the observation that the analogue of (6.1) for a general convex function g is

$$g(u) = \max_v [g(v) + (u - v)\, g'(v)], \tag{1}$$

a convex curve is the envelope of its tangents.

From this follows

$$J(u) = \int_0^T (u'^2 + 2g(u))\, dt \geqslant \int_0^T [u'^2 + 2g(v) + 2(u - v)\, g'(v)]\, dt \tag{2}$$

for all v restricted by the conditions $v(0) = c$, $v' \in L^2(0, T)$. Hence,

$$\min_u J(u) \geqslant \min_u \left[\int_0^T (u'^2 + 2ug'(v))\, dt + \int_0^T [2g(v) - 2vg'(v)]\, dt\right]. \tag{3}$$

The minimization with respect to u on the right-hand side may be readily carried out since the Euler equation has the form

$$u'' - g'(v) = 0, \tag{4}$$

$$u(0) = c, \qquad u'(T) = 0.$$

Write

$$K_1(v) = \min_u \left[\int_0^T (u'^2 + 2ug'(v))\, dt\right]. \tag{5}$$

Then

$$\min_u J(u) \geqslant \max_v \left[K_1(v) + \int_0^T [2g(v) - 2vg'(v)]\, dt\right]. \tag{6}$$

Once again a direct calculation shows that equality holds. This method was first given in [14].

8. Perturbation Techniques

One advantage of the foregoing technique is that we can if we wish apply it to some selected part of the functional. Suppose, for example, that

$$g(u) = u^2 + u^4, \tag{1}$$

with $|c| \ll 1$ so that u^4 may be regarded as a perturbation term. We write

$$J(u) = \int_0^T (u'^2 + u^2 + u^4)\, dt \geqslant \int_0^T (u'^2 + u^2 + 4uv^3 - 3v^4)\, dt, \tag{2}$$

applying (7.1) to the function u^4. Thus,

$$\min_u J(u) \geqslant \min_u \left[\int_0^T (u'^2 + u^2 + 4uv^3)\, dt - 3\int_0^T v^4\, dt\right]. \tag{3}$$

The minimization of the quadratic functional is readily performed [4], yielding

$$\min_u J(u) \geqslant \max_v \left[K_2(v) - 3\int_0^T v^4\, dt\right]. \tag{4}$$

Once again, a direct calculation shows that equality is attained. Since $v = u$ for the maximizing v, we suspect that simple approximations for u, say that obtained from taking $g(u) = u^2$, will yield good initial approximations.

9. Use of Convexity of h

Let us now turn to the more general case where $h \neq 0$,

$$f_T = 2g(c) - \frac{f_c^2}{2}, \qquad f(c, 0) = h(c), \tag{1}$$

adding the assumption that h is convex. Turning to Section 6, we see that we can follow the method outlined there at the expense of minimizing the functional

$$J_2(u) = \int_0^T (2g(u) - uv')\, dt + h(u(T)), \tag{2}$$

over functions u such that $u(0) = c$, $u' \in L^2(0, T)$.

Alternatively, we can use the convexity of h,

$$h(u(T)) = \max_b [h(b) + h'(b)(u(T) - b)], \tag{3}$$

writing

$$\min_u J(u) \geqslant \max_b \min_u [h(b) + h'(b)(u(T) - b) + \cdots], \tag{4}$$

and so forth.

10. $2g(c) = c^2$, h Convex

The foregoing method is particularly attractive when $2g(c) = c^2$. The partial differential equation is

$$f_T = c^2 - \frac{f_c^{\,2}}{4}, \qquad f(c, 0) = h(c), \tag{1}$$

and

$$J(u) = \int_0^T (u'^2 + u^2)\, dt + h(u(T)). \tag{2}$$

Proceeding as above, we obtain the result

$$\min_u J(u) \geqslant \max_b \left[\min_u \left[\int_0^T (u'^2 + u^2)\, dt + u(T)\, h'(b)\right] + h(b) - bh'(b)\right]. \tag{3}$$

The minimization with respect to u may be readily carried out, yielding

$$\min_u J(u) \geqslant \max_b [q_2(c, h'(b)) + h(b) - bh'(b)]. \tag{4}$$

Here q_2 is a quadratic in c and $h'(b)$ with coefficients depending on T, and the desired value of b is $u(T)$. Again, equality holds.

References

1. R. Bellman. "Dynamic Programming." Princeton Univ. Press, Princeton, New Jersey, 1957.
2. R. Bellman. Functional equations in the theory of dynamic programming—V: Positivity and Quasilinearity. *Proc. Nat. Acad. Sci. USA.* **41** (1955), 743–746.
3. R. Bellman and R. Kalaba. "Quasilinearization and Nonlinear Boundary-value Problems." American Elsevier, New York, 1965.
4. R. Bellman. "Introduction to the Mathematical Theory of Control Processes." Volume I. Academic Press, New York, 1968.
5. R. Bellman. "Introduction to the Mathematical Theory of Control Processes." Vol. II. Academic Press, New York (to appear).

6. E. F. BECKENBACH AND R. BELLMAN. "Inequalities," Springer-Verlag, Berlin, 1961.
7. R. KALABA. On Nonlinear differential equations, the maximum operation, and monotone convergence. *J. Math. Mech.* **8** (1959), 519–574.
8. E. D. CONWAY AND E. HOPF. Hamilton's theory and generalized solutions of the Hamilton-Jacobi equations. *J. Math. Mech.*, Vol. 13, 1964, pp. 939–986.
9. E. HOPF. Generalized solutions of Nonlinear equations of the first order. *J. Math. Mech.* **14** (1965), 951–973.
10. F. CALOGERO. "Variable Phase Approach to Potential Scattering." Academic Press, New York, 1967.
11. R. BELLMAN. Upper and Lower bounds for the solution of the matrix Riccati equation. *J. Math. Anal. Appl.* **17** (1967), 373–379.
12. S. KAPLAN. Canonical and involutory transformations of variational problems involving higher derivatives. *J. Math. Anal. Appl.* **22** (1968), 45–53.
13. W. FENCHEL. On conjugate convex functions. *Canadian J. Math.* **1** (1949), 73–77.
14. R. BELLMAN. Quasilinearization and upper and lower bounds for variational problems. *Q. Appl. Math.* **19** (1962), 349–350.